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# THE POINT-CIRCLE VORTEX†

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A new kind of vortex structure—the point-circle vortex (PCV)—is investigated. It can preserve its symmetry as it evolves with time. A leap-forging PCV is discussed together with its randomization and collapse.

The evolution of a system of $N$ point vortices is described by the equations [1]

$$\frac{d\bar{z}_j(t)}{dt}=\frac{1}{2\pi i}\sum_{k\neq j}^{N}\frac{\gamma_k}{z_j-z_k}\quad (j=1,\ldots,N) \tag{1}$$

which admit of the Kirchhoff invariants

$$I_1=\sum_j^N\gamma_j z_j,\quad I_2=\sum_j^N\gamma_j|z_j|^2,\quad I_3=\sum_k^N\sum_{j\neq k}^N\gamma_k\gamma_j\ln|z_k-z_j| \tag{2}$$

where $z_j$ is the complex coordinate of a vortex, $\gamma_j$ is its circulation, and the bar denotes conjugation. Here and henceforth summation begins with the term corresponding to the index 1.

We consider a PCV—a system point vortices located on $n$ concentric circles so that the circulations $\Gamma_\alpha$ ($\alpha=1,\ldots,n$) of all the vortices situated on the $\alpha$th circle are the same, they are separated by equal angular intervals $2\pi/N$, and the number of vortices $N$ on each circle is the same (Fig. 1, with the vortices shown by small circles). For $N=1$ the system of $n$ point vortices is not a PCV.

A general remark should be made about investigations of the stability of symmetric clusters. One can either investigate the symmetry breaking of the system as a whole, or one can fix the symmetry as an exact feature of the problem, and investigate the instability of the resulting product. This splitting of the problem provides a single approach to understanding the mechanism of instability for vortex structures.

In the case of a PCV one can follow the behaviour of all $nN$ vortices simultaneously. Then the resulting "cloud" of vortices demonstrates symmetry instability similar to the Helmholtz instability [2] inherent in contact-discontinuities: the PCV is immediately disrupted [3]. In the other case, choosing a single vortex on each circle and symmetrically fixing the positions of the remaining vortices, one can follow the instability of $n$ vortices. The mechanism of this instability, which is significantly different from the Helmholtz instability, is the subject of this paper.

Thus the PCV will rotate about its centre $r=0$ preserving its symmetry, i.e. equal angular distances between vortices lying on any chosen circle. The evolution of the PCV is governed by the following parameters: $N, n, \Gamma_\alpha, r_\alpha(0), \theta_\alpha(0)$ where $r_\alpha(0)$ and $\theta_\alpha(0)$ are the values of the polar coordinates of a vortex chosen arbitrarily on the $\alpha$th circle at initial time $t=0$. Of these $3n+2$ parameters $\theta_1(0)$ is unimportant, and hence the number of governing parameters is $3n+1$. The value of $r_1(t)$ could be equal to zero.

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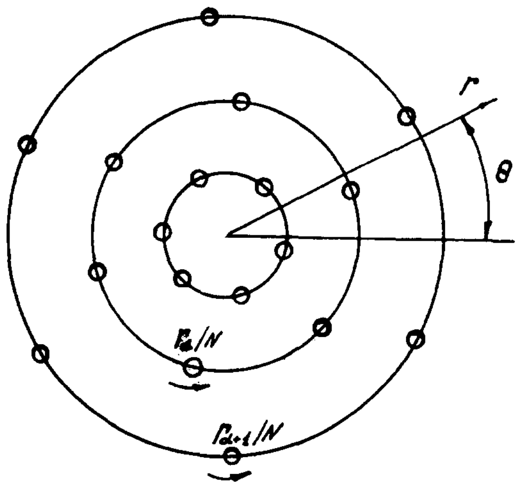


Fig. 1.

With suitably chosen $\Gamma_\alpha$ passage to the limit $N \to \infty$ produces a system of concentric vortex sheets. Unlike a contact discontinuity the Bernoulli number at a vortex sheet is constant. Hence in the limiting stationary vortex sheet case that we are considering, their velocity will vary in direction, but not in magnitude. From this condition we conclude that the passage to the limit of vortex sheets is correct if the circulation of the vortices is specified by the recurrence relations

$$\Gamma_\alpha = -2\sum_k^{\alpha-1}\Gamma_k, \quad \Gamma_1 \neq 0, \quad r_1 = 0$$

From the definition of the PCV the coordinator of the $j$th vortex located on the $\alpha$th circle is given by the relation

$$z_\alpha^j = r_\alpha \exp[i(\theta\alpha + 2\pi j / N)]$$

Decomposing the sums in (1) into two parts, one of which is over vortices lying on the same circle, and the other is over vortices located on the other circles, we obtain

$$\frac{d\bar{z}_\alpha^j(t)}{dt} = \frac{1}{2\pi i N}\left(\Gamma_\alpha \sum_{k\neq j}^{N} \frac{1}{z_\alpha^j - z_\alpha^k} + \sum_{\beta\neq\alpha}^{n} \Gamma_\beta \sum_k^N \frac{1}{z_\alpha^j - z_\beta^k}\right) \tag{3}$$

To fix our ideas we shall follow the vortex numbered $N$. The series over $k$ in (3) are easily summed

$$\sum_k^{N-1} \frac{1}{1-\exp(ik\varphi)} = \frac{N-1}{2}$$

$$\sum_k^N \frac{1}{1 - x_{\alpha\beta}\exp[i(k\varphi + \theta_{\alpha\beta})]} = \frac{1}{1 - x_{\alpha\beta}^N \exp(iN\theta_{\alpha\beta})}$$

$$\varphi = 2\pi / N, \quad x_{\alpha\beta} = r_\beta / r_\alpha, \quad \theta_{\alpha\beta} = \theta_\beta - \theta_\alpha, \quad \alpha, \beta = 1, \ldots, n$$

After reduction system (3) can be represented in the form

$$\frac{dr_\alpha^2}{dt} = \frac{1}{2\pi}\sum_{\beta\neq\alpha}^{n} \Gamma_\beta \frac{\sin N\theta_{\alpha\beta}}{\Delta_{\alpha\beta}}$$

$$\frac{d\theta_\alpha}{dt} = \frac{1}{4\pi r_2^\alpha}\left(\frac{N-1}{N}\Gamma_\alpha + \sum_{\beta\neq\alpha}^{n}\Gamma_\beta \frac{x_{\alpha\beta}^{-N} - \cos N\theta_{\alpha\beta}}{\Delta_{\alpha\beta}}\right) \tag{4}$$

$$\Delta_{\alpha\beta} = (x_{\alpha\beta}^{N} + x_{\alpha\beta}^{-N})/2 - \cos N\theta_{\alpha\beta}$$

The first invariant $I_1$, expressing the law of conservation for the "centre of gravity" of the system, is equal to zero for the PCV. The second and third invariants, expressing the conservation of dispersion and energy [1], have the form

$$I_2 = \sum_{\alpha}^{n}\Gamma_\alpha r_\alpha^2$$

$$I_3 = \sum_{\alpha}^{n}\xi_{\alpha\alpha}\left(\frac{N-1}{2}\ln\eta_\alpha + \ln N\right) + \frac{1}{2}\sum_{\beta\neq\alpha}^{n}\sum_{\alpha}^{n}\xi_{\alpha\beta}(N\ln\eta_\alpha + \ln\Delta_{\alpha\beta}) \tag{5}$$

$$\xi_{\alpha\beta} = \Gamma_\alpha\Gamma_\beta / N, \quad \eta_\alpha = \frac{I_2}{\Gamma_\alpha + \sum_{\delta\neq\alpha}^{n}\Gamma_\delta x_{\alpha\delta}^2}$$

We will consider some special cases.

*A unary PCV* (n = 1). The solution is obtained in closed form

$$r = \text{const}, \quad \frac{d\theta}{dt} = \frac{N-1}{4\pi r^2 N}\Gamma$$

*A binary PCV* (n = 2). With the help of the second and third invariants (5) one can reduce the order of system (4). The evolution of the PCV is described by one differential and one algebraic equation in the variables $x = x_{12}$, $\theta = \theta_{12}$

$$\frac{dx^2}{dt} = -\frac{I_2 \sin N\theta}{2\pi\eta_1^2\Delta_{12}}$$

$$I_3 = \xi_{12}\ln\Delta_{12} + \tfrac{1}{2}[(N-1)\xi_{11} + N\xi_{12}]\ln\eta_1 +$$

$$+\tfrac{1}{2}[(N-1)\xi_{22} + N\xi_{12}]\ln\eta_1 x^2 + (\xi_{11} + \xi_{22})\ln N \tag{6}$$

A simple graphical method of investigating the evolution of system (6) makes use of the phase trajectories $I_3(r_1^2, \theta) = \text{const}$. In Fig. 2 we show a typical phase trajectory topology ($N = 5$, $I_2 = 2$, $\Gamma_1 = \Gamma_2 = 1$). The rectangle $0 \leqslant r_1^2 \leqslant 1$, $0 \leqslant \theta \leqslant \pi/N$ was chosen because the family of trajectories in the interval $1 \leqslant r_1^2 \leqslant 2$ and trajectories in the rectangle $0 \leqslant r_1^2 \leqslant 1$, $\pi/N \leqslant \theta \leqslant 2\pi/N$ are symmetric about the line $r_1^2 = 1$ to the trajectories shown in Fig. 2. Furthermore, phase trajectories in the $0 \leqslant \theta \leqslant 2\pi/N$ sector are periodically repeated in the sectors $2\pi k/N \leqslant \theta \leqslant 2\pi(k+1)/N$, $k = 1, \ldots, N-1$.

The phase plane is divided into two domains with fundamentally different regimes. The boundary between these domains is shown by the dashed line. In domain 1, which lies below the dashed line in Fig. 2, only weak interaction is observed and the vortex circles, rotating with different velocities, do not pass through one another. In domain 2, which lies above the dashed line, a strong ("leap-frog") interaction is observed, with the vortex circles periodically passing through one another. The equation for the dashed line is obtained from the relation

$$I_3(x,\theta) = I_3(I_2/2, \ \pi/N) \tag{7}$$

In the case $I_2 = 2$, $\Gamma_1 = \Gamma_2 = 1$ shown in Fig. 2 the minimum value $x = x_{\min}$ is reached at $\theta = 0$; from relation (7) it follows that

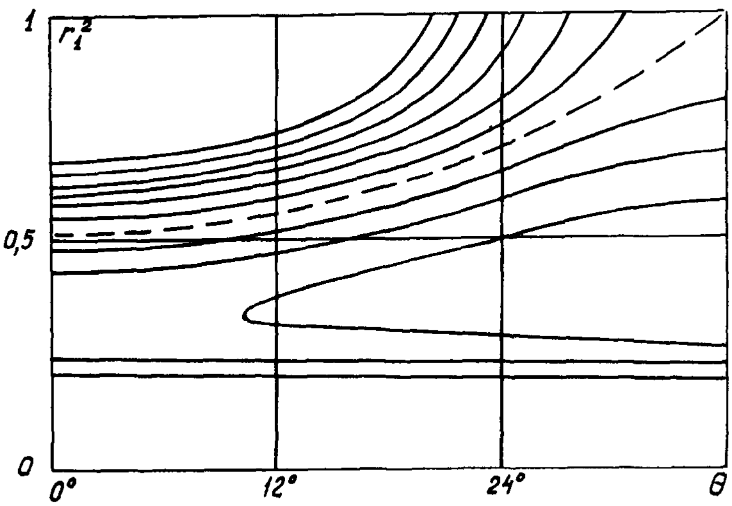


Fig. 2.

$$2^{2N-3} x_{\min}^{N-1} (x_{\min}^{N} - 1)^2 = (1 + x_{\min}^2)^{2N-1}$$

As $N \to \infty$ we have $x_{\min} = 1 - a/N$ where $a = \ln(3+\sqrt{8})$.

In Fig. 3 we show an example of the time dependence of the radii $r_1(t)$, $r_2(t)$ of the circles for the strongly interacting case when $N=5$, $\Gamma_1 = 1$, $\Gamma_2 = 2$.

Point vortices are a computationally necessary mathematical idealization of actual vortex fields: three-dimensional vortices or lines of tangential velocity discontinuity. Such a discretization adds new non-intrinsic properties, one of which is the strong interaction. Hence in numerical calculations of the evolution of a spiral vortex sheet (Fig. 4) the discrete vortex method should avoid too coarse a decomposition step, in order not to end up in the strong interaction domain, i.e. with a non-physical solution. Using the results of the above model problem one can make recommendations on the choice of step size relative to the distance between the loops of the spiral.

*Triple PCV* (n = 3). Using the invariants $I_2$ and $I_3$ one can reduce the order of system (4) to four, but it remains complicated. Hence this vortex ensemble was investigated numerically, using a fourth-order Runge–Kutta algorithm.

Analysis of the motion of this system naturally revealed many more interaction situations than for the binary PCV. We will list them.

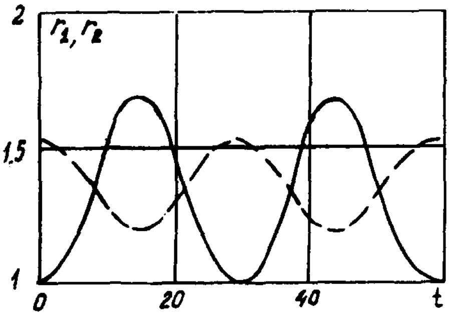


Fig. 3.

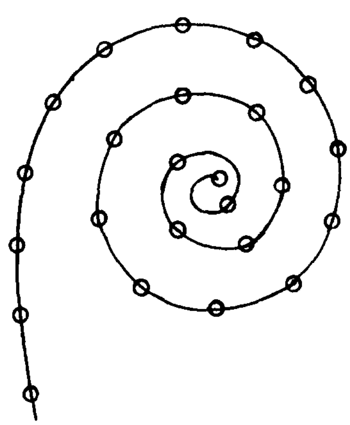

Fig. 4.

1. *Weak interaction*. This is similar to the binary PCV case (Fig. 5).

2. *Pairwise strong interaction*. In this case vortices from two circles strongly interact with each other and change places. The third vortex circle may (Fig. 6) or may not (Fig. 7) strongly interact with the two "coupled" circles.

3. *Triple strong interaction* (Fig. 8).

4. *Chaotic interaction* According to statistical mechanics [5, 6], chaos can appear in a system described by three or more non-linear differential equations [7]. One possible scenario for the onset of chaotic behaviour is the appearance of an $\varepsilon$-layer on the separatrices in phase space [8]. Numerical investigation of system (4) revealed a manifold of chaotic regimes. In Fig. 9 we show a case when the system behaved randomly at first, but after some time fell into a strong periodic interaction domain, and such scenarios ("relaminarization") are frequently encountered. An intermittency regime is also observed in which the system alternates between a chaotic regime and a pairwise interaction regime (Fig. 10). It should be noted that the system is very sensitive to the initial conditions.

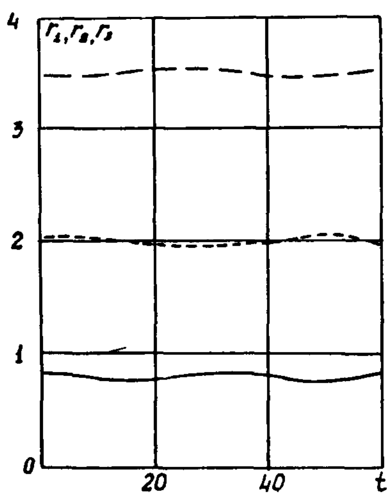


Fig. 5.

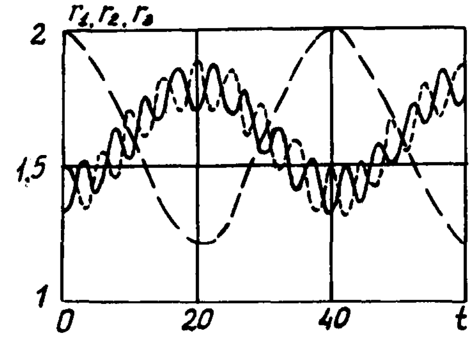


Fig. 6.

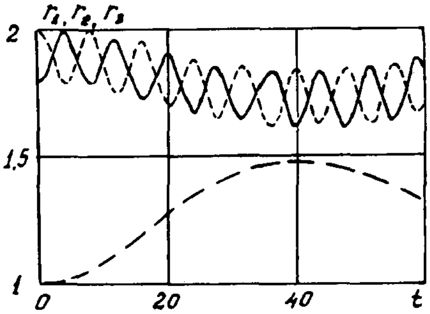


Fig. 7.

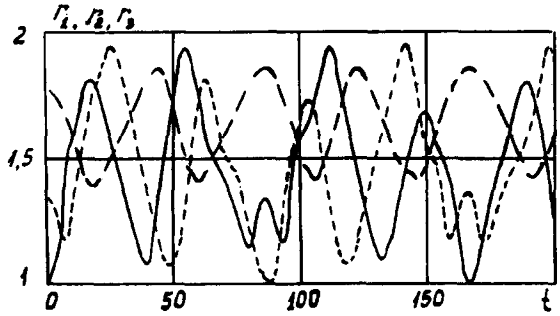


Fig. 8

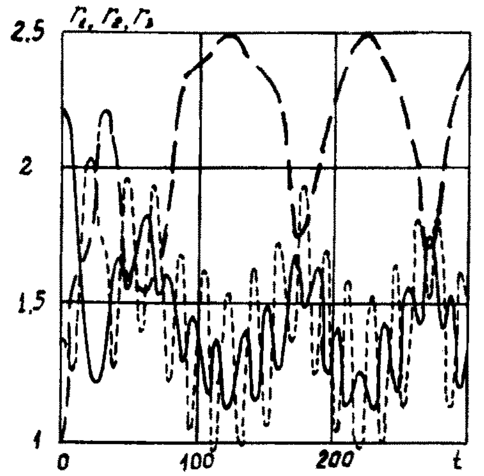


Fig. 9.

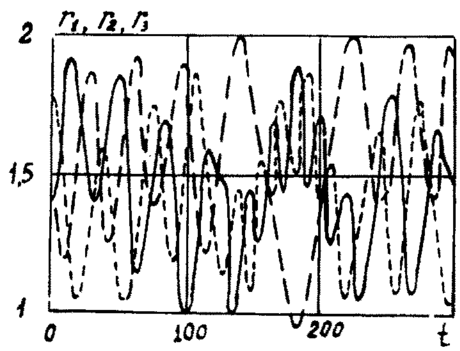


Fig. 10.

We know that a suitably positioned system of three or more vortices can undergo self-similar collapse [9]. Such a collapse has not been revealed in numerical observations of PCVs. It would appear to be unstable, and when three vortices approach one another the influence of the remaining vortices should be considered as a non-self-similar perturbation of the self-similar collapse process.

Unlike the collapse process, which is completely deterministic if initial vortex interactions consistent with it are specified, the break-up of a single point vortex when there are no external influences cannot occur by natural causally determined processes: there is no initial start time for the break-up or for its other characteristics. Indeterminancy in the break-up of a vortex leads to indeterminancy in the dynamics of vortex systems and, possibly, in ideal fluid dynamics in general. Such an unusual mechanism for the "arrow of time" has no analogue in particle dynamics.

Problems of the existence and stability of stationary and uniformly rotating clusters of point vortices have been discussed in [10].

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